THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 4

1 Let L be the length of the contour C. Given $z \in \Omega$, define $r = \min_{w \in C} |z - w|$ to be the distance between the point z and the contour C. Then for any Δz with $|\Delta z| < r/2$, we have $|s - (z + \Delta z)| \geq |s - z| - |\Delta z| \geq r/2$. Note that

$$\frac{1}{\Delta z}\left(\int_C \frac{f(s)}{s-(z+\Delta z)}ds - \int_C \frac{f(s)}{s-z}ds\right) = \int_C \frac{f(s)}{(s-z)(s-(z+\Delta z))}ds$$

Since $f(s)$ is a continuous function, $M = \max_{s \in \bar{\Omega}} f(s)$ exists. From this we have

$$\begin{aligned}
&\left|\frac{1}{\Delta z}\left(\int_C \frac{f(s)}{s-(z+\Delta z)}ds - \int_C \frac{f(s)}{s-z}ds\right) - \int_C \frac{f(s)}{(s-z)^2}ds\right| \\
=&\left|\int_C \frac{f(s)}{(s-z)(s-(z+\Delta z))}ds - \int_C \frac{f(s)}{(s-z)^2}ds\right| \\
=&\left|\int_C \frac{f(s)\Delta z}{(s-z)^2(s-(z+\Delta z))}ds\right| \\
\leq&\left|L \times \frac{M}{r^2(r/2)}\Delta z\right| \xrightarrow{\Delta z \to 0} 0
\end{aligned}$$

This gives the desired result.

2 Define a function $g(z) = \exp(f(z))$. Since $f(z)$ is entire, $g(z)$ is also entire. Furthermore,

$$|g(z)| = e^{Re(f(z))} \leq e^{u_0}$$

Hence $g(z)$ is an entire and bounded function. By Liouville's theorem, $g(z)$ must be constant. Since $f(z)$ is continuous and $g(z)$ is a constant function, we must have $f(z) =$ constant.

3 Since $f(z)$ is entire, we have $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \int_{|z|=r} \frac{f(s)}{s^{n+1}}ds$ and r is arbitrary positive real number. In particular, since $|f(z)| \leq M|z|$, for $n \geq 2$ we have

$$a_n = \int_{|z|=r} \frac{f(s)}{s^{n+1}}ds \leq \text{length of the contour} \times \frac{Mr}{r^{n+1}} = 2\pi r \frac{Mr}{r^{n+1}} = \frac{2\pi M}{r^{n-1}} \xrightarrow{r \to \infty} 0$$

Hence we have $a_n = 0$ for $n \geq 2$ and $f(z) = a_0 + a_1 z$. Since $|f(z)| \leq M|z|$, $|a_0| = |f(0)| \leq M(0) = 0$. So we have $f(z) = a_1 z$ for some $a_1 \in \mathbb{C}$.

4 First, by Cauchy Integral formula, we have

$$\int_{|z|=1} \frac{e^{az}}{z}dz = 2\pi i(e^{a(0)}) = 2\pi i$$

On the other hand,

$$\int_{|z|=1} \frac{e^{az}}{z}dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} ie^{i\theta}d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta + ai\sin\theta}d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta) + i\sin(a\sin\theta))d\theta$$

This implies

$$2\pi i = \int_{|z|=1} \frac{e^{az}}{z} dz = -\int_{-\pi}^{\pi} e^{a\cos\theta} \sin(a\sin\theta) d\theta + i \int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta$$

By comparing the imaginary parts on both sides, we have

$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta = 2\pi$$

Since

$$\begin{aligned}\int_{-\pi}^{0} e^{a\cos\theta} \cos(a\sin\theta) d\theta &= \int_{\pi}^{0} e^{a\cos(-\theta)} \cos(a\sin(-\theta)) d(-\theta) \\ &= \int_{0}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta,\end{aligned}$$

we have

$$\int_{0}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta = \pi$$

5 Cauchy Integral formula states that for $n = 0, 1, 2, \ldots$, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|s|=R} \frac{f(s)}{(s-z)^{n+1}} ds$$

By the analyticity of the integrand, we also have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|s|=R} \frac{f(s)}{(s-z)^{n+1}} ds = \frac{n!}{2\pi i} \int_{|s-z|=R-|z|} \frac{f(s)}{(s-z)^{n+1}} ds$$

Therefore we have

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_{|s-z|=R-|z|} \frac{f(s)}{(s-z)^{n+1}} ds \right| \leq 2\pi(R-|z|) \times \frac{n!}{2\pi} \frac{M}{(R-|z|)^{n+1}} = \frac{n!M}{(R-|z|)^n}$$

6 Note that since the function $\frac{1+z}{1-z}$ is not well-defined at $z = 1$, f is not analytic at $z \neq 1$. Furthermore, for $z \neq 1$,

$$\begin{aligned} & \frac{1+z}{1-z} = -r, \text{ where } r \geq 0 \\ \iff\quad & 1 + z = -r + rz \\ \iff\quad & z = \frac{r+1}{r-1} = 1 + \frac{2}{r-1} \\ \iff\quad & z \in (-\infty, -1] \cup (1, \infty) \end{aligned}$$

This implies that the function $\tanh^{-1}(z)$ is analytic on $\mathbb{C} \backslash \{z \in \mathbb{C} : z = x \in (-\infty, -1] \cup [1, \infty)\}$.

7 (a)

$$\begin{aligned} f(z) =& \frac{1}{(z-1)(z-2)} \\ =& \frac{1}{1-z} - \frac{1}{2} \frac{1}{1-\frac{z}{2}} \\ =& \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} \\ =& \sum_{n=0}^{\infty} (1 - \frac{1}{2^{n+1}}) z^n \end{aligned}$$

(b)

$$\begin{aligned}
f(z) =& \frac{1}{(z-1)(z-2)} \\
=& \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\
=& -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) \\
=& -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \\
=& -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}
\end{aligned}$$

(c)

$$\begin{aligned}
f(z) =& \frac{1}{(z-1)(z-2)} \\
=& -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) + \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) \\
=& -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} \\
=& \sum_{n=0}^{\infty}\frac{2^n-1}{z^{n+1}} \\
=& \sum_{n=1}^{\infty}\frac{2^n-1}{z^{n+1}}
\end{aligned}$$